Vertical motion of a floating sphere in a sine-wave sea

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The vertical motion (heave) of a freely floating sphere is studied under the action of incident sine waves. Forced heave is defined as the vertical motion of sphere in still water, while free heave is defined as the vertical motion of sphere in the sine-wave sea. The total velocity potential describing the motion is decomposed into three terms: incident wave potential, diffracted wave potential (potential as if the sphere were fixed in sine-wave sea), and forced heave potential (potential as if the sphere were in forced vertical motion in still water). The forced heave and diffraction problems are solved separately. The linearized equation of motion is then used to effect the synthesis of the free motion via two unknowns: amplitude of vertical displacement and phase difference.

Both the radiation and diffraction problems are solved by expansions of non-orthogonal functions (wave-free potentials). These functions are trivial solutions of the Sommerfeld radiation condition in the sense that they attenuate faster than $O(r^{-\frac{1}{2}})$ and it is necessary to add multipole terms which have the proper behaviour at infinity. Infinite systems of linear equations are obtained for the unknown expansion coefficients and the unknown source strengths of the multipole. The added mass, damping coefficient and wave-making coefficient in forced heave are studied as well as the force on the fixed sphere. In addition the added mass and damping coefficient in free heave are obtained.

The horizontal motion (surge) of the freely floating sphere can be obtained by methods similar to those employed for the vertical motion. The author is presently working on this problem. Surge and heave are not independent motions but can be treated separately because of the symmetry of the sphere.

1. Introduction

The rigorous solution of the motions of a freely floating rigid body in surface waves is an extremely difficult one. At present, theoretical attempts are based upon the linearized equations of motion of a perfect fluid. Even after this simplification one is still left with a complex problem.

At this point two alternatives are open: first, the use of some physical hypothesis to further simplify the problem, and secondly, an attempt to solve the problem rigorously (for some simple body). Let us take the first alternative and examine its implications. Most attempts to determine the motions have employed the Froude–Krylov hypothesis which states that at each point of the immersed surface of the rigid body the same pressure acts as would occur if the body were not there, the reaction of the rigid body upon the seaway being neglected. Instead of attempting the solution of a complicated hydrodynamic problem, the Froude–Krylov hypothesis circumvents this by assuming that the effect of the waves upon the object is contained in two empirical terms involving the amplitude η of the response of the object. For heaving motion they are (*a*) $M\,d^2\eta/dt^2$, where M is a constant called the added mass; and (*b*) $N\,d\eta/dt$, where N is a constant called the damping factor. These two 'constants' are determined experimentally by forced oscillation of the rigid body in calm water; a further assumption being that these 'constants' are independent of the frequency of oscillation. Finally, the forcing function is taken as the pressure due to the incident wave as if the vessel were not there. The resultant second-order differential equation is then solved.

Those familiar with diffraction theory will notice the similarity between the Froude–Krylov hypothesis and Kirchhoff's scalar theory of diffraction as regards the assumptions concerning the distortion of the incident wave field. This empirical method, although capable of powerful results, cannot by its very nature lead to a rational hydrodynamic theory of rigid body motion.

The second alternative does not depend upon experimental data and yields solutions complete within themselves. There are two types of problems to be considered, radiation problems and diffraction problems. The radiation problems are concerned with the determination of the fluid motion produced by periodically forced oscillations of an object in the free surface. Ursell (1949, 1953, 1954, 1957) in a series of important papers has attacked and solved the two-dimensional problem of the forced periodic heaving of an infinitely long circular cylinder semi-immersed in the free surface, while Havelock (1955) has formally solved the three-dimensional problem of the forced periodic heaving of a semi-immersed sphere. In the radiation problems considered by these investigators, the normal component of the velocity of the object is known *a priori* by virtue of specifying the heaving velocity.

The second type of problem, the diffraction problem, is more complex. In diffraction problems we assume that the free surface of the fluid supports a two-dimensional monochromatic progressive wave, and we are required to compute the scattering of the incident wave upon the introduction of an obstacle (rigid or freely floating), while simultaneously determining the motion of the obstacle if it is freely floating. Only the steady-state case is considered. In the case where there is a freely-floating object the problem becomes extremely difficult in that the normal component of the velocity of the object is not known but must be determined as part of the solution.

The two-dimensional problems possess the peculiarity that the free surface boundary condition at low heave frequency ($\beta \sim 0$) degenerates in such a manner that the problem becomes indeterminate. This indeterminateness is manifested by the occurrence of logarithmic terms in β, as a consequence the added mass coefficient becomes infinite as β approach zero. In contrast the three-dimensional problem is determinate at $\beta = 0$.

The simplest case of a three-dimensional situation is that of a floating sphere semi-immersed in the free surface and undergoing heave motion due to the resultant action of the incident sine waves. Our problem is to determine the motion given only the geometry of the sphere and the wavelength of the incident wave under the tacit restriction to steady-state motions only.

The sphere, being a highly degenerate geometric object, can be shown to have only two possible degrees of freedom in the linearized theory; they are vertical motion (heave) and horizontal motion (surge). It is a rigorous consequence of linearized theory that these two motions are uncoupled (John 1949) because of the symmetry of the sphere. This is the justification for treating heave only. The problem of surge can be treated by analogous methods and is presently being investigated by the author.

For reasons to be shown in § 3, it is necessary to solve both radiation (forced heave) and diffraction aspects of the problem. The final solution of the freely floating sphere is obtained by synthesis of the radiation and diffraction problems through the equation of motion.

The present paper is based in part on a condensed version of an internal report (Barakat 1960) which contains the detailed analysis and numerical results. This report is available to interested readers.

In a future paper the author will discuss the vertical motion of a floating sphere in a stochastic seaway.

2. Formulation of the problem

The linearized theory depends upon obtaining the potential function for the description of the motion. The potential function $\psi(r,\theta,\phi,t)$ is harmonic in the interior of the fluid and satisfies certain boundary conditions. Taking $\psi(r,\theta,\phi,t)$ to be harmonically varying in time, we have

$$\psi(r,\theta,\phi,t) = \mathrm{Re}\,[W(r,\theta,\phi)\,e^{-i\omega t}]. \tag{2.1}$$

$W(r,\theta,\phi)$ satisfies

(A) $\nabla^2 W = 0$ (in fluid),

(B) $\dfrac{\partial W}{\partial z} + k_0 W = 0$ (on free surface) $k_0 = \omega^2/g$,

(C) $W_n = V_n$ (on immersed surface or sphere),

where V_n is the normal velocity of the sphere. If the sphere were in forced motion it would be possible to specify V_n; however, when a sphere is freely floating we cannot specify V_n but must obtain it as part of the complete solution. We also demand the derivatives of W with respect to z be bounded and converge to zero as z approaches infinity.

There is still another condition which must be specified at infinity; namely, that the scattered or radiated waves (W_2) behave as outgoing progressive waves. This restriction upon $W_2(r,\theta,\phi)$ is the radiation condition of Sommerfeld which requires that (see John 1950)

(D) $\lim\limits_{r\to\infty} r^{\frac{1}{2}}[\partial W_2/\partial r - ikW_2] = 0$

uniformly in r and θ with fixed positive constant k. We can look upon this constraint as the assertion that we exclude the superposition of free waves upon the waves generated by the interaction of the incident waves and the obstacle. John

(1950) has shown by an energy conservation argument that the wave motion due to a finite obstacle should decay as

$$O[(x^2+y^2)^{-\frac{1}{4}}] = O(r^{-\frac{1}{2}}),$$

as r tends to infinity along the free surface. This argument is equivalent to the statement that the wave function W_2 which satisfies the Sommerfeld radiation condition must attenuate as $O(r^{-\frac{1}{2}})$. The wave-free potentials (§ 4) attenuate faster than $O(r^{-\frac{1}{2}})$, in fact at least as $O(r^{-2})$. They simply attenuate too rapidly to transport energy to infinity *at the required rate* as demanded by the radiation condition.

If we were to attack the problem as an initial value problem and allow the time to approach infinity, then one would need to impose only boundedness conditions at infinity. Although in principle the sphere could be handled in this manner, it must be confessed that at the present time this attack presents almost insurmountable analysis.

3. Method of solution

The linearized equation of motion for the vertical displacement (heave) of a semi-immersed sphere in the steady-state is given by (John 1949)

$$\pi(\tfrac{2}{3}\beta+1)\,\eta_0\,e^{i\epsilon} = -i\frac{\omega}{g}\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} W(a,\theta,\phi)\sin\theta\,\cos\theta\,d\phi, \tag{3.1}$$

where η is the vertical displacement of the sphere and

$$\eta = \eta_0\,e^{-i\omega t+i\epsilon}, \tag{3.2}$$

where ϵ is an unknown phase angle which is to be determined as part of the solution. The dimensionless parameter $\beta = k_0 a$ is essentially a Froude number.

We *assume* the total velocity potential W to be given by

$$W = W^i + W^d + W^f e^{+i\epsilon}, \tag{3.3}$$

where W^i = potential due to incident sine waves, W^d = potential due to scattered waves from sphere if it were assumed rigid, and W^f = potential due to forced vertical motion of sphere. Now

$$\frac{\partial W}{\partial r} = \frac{\partial W^i}{\partial r}+\frac{\partial W^d}{\partial r}+e^{i\epsilon}\frac{\partial W^f}{\partial r} = V_n \quad (r=a), \tag{3.4}$$

but by construction (or definition)

$$\frac{\partial W^i}{\partial r}+\frac{\partial W^d}{\partial r} = 0 \quad (r=a), \tag{3.5}$$

leaving

$$e^{i\epsilon}\frac{\partial W^f}{\partial r} = V_n. \tag{3.6}$$

In (3.6) V_n is given. In this manner we construct a velocity potential of the motion which satisfies the boundary conditions on the body and behaves properly at infinity. This decomposition is a generalization of the methods used by Ursell (1949) in his solution of the forced heaving of a circular cylinder.

First we solve for the diffraction of water waves from a fixed sphere and add to it the motion due to the forced heaving of the sphere. The solution of these two problems yields two unknowns, the phase of the motion of the sphere and the amplitude of heave. We now use the linearized equation of heave (3.1) in order to compute these two quantities explicitly.

4. Wave-free potentials

These are two combinations of basic eigenfunctions which satisfy the free surface condition and Laplace's equation in spherical co-ordinates:

$$\left.\begin{aligned} M_{2n}^{2m} &= \left[\frac{P_{2n}^{2m}(\mu)}{r^{2n+1}} + \frac{k_0 P_{2n-1}^{2m}(\mu)}{2(n-m)\, r^{2n}}\right] \cos 2m\phi \quad \begin{pmatrix} n = 1, \dots, \infty \\ m = 0, \dots, n-1 \end{pmatrix}, \\ N_{2n}^{2m-1} &= \left[\frac{P_{2n+1}^{2m-1}(\mu)}{r^{2n+2}} + \frac{k_0 P_{2n}^{2m-1}(\mu)}{2(n-m+1)\, r^{2n+1}}\right] \cos (2m-1)\,\phi \quad \begin{pmatrix} n = 1, \dots, \infty \\ m = 1, \dots, n \end{pmatrix}. \end{aligned}\right\} \tag{4.1}$$

These two particular combinations M_{2n}^{2m} and N_{2n}^{2m-1} of non-orthogonal harmonic functions are termed wave-free potentials generalizing Havelock's definition (1955). The wave-free potentials represent local oscillations of the free surface in the immediate vicinity of the sphere which decay very rapidly as we move away from the sphere. Both of the wave-free potentials are $O(r^{-2})$ or greater. These functions trivially satisfy the radiation condition which demands the functions decay as $O(r^{-\frac{1}{2}})$; the wave-free potentials simply decay too rapidly at infinity.

5. Multipole terms

Since the wave-free potentials are trivial solutions of the radiation condition, it is necessary to add a singular solution (of the appropriate multiplicity) of Laplace's equation which satisfies the free surface condition and behaves properly at infinity.

There are several methods available for the explicit calculation of the singularities; however, the method due to Ursell (1950) seems to be the most elegant and powerful at present. Using a variant of Ursell's method, Thorne (1953) has shown that the multipole source G of unit strength at $(0, 0, f)$ in *infinitely deep water* is given by

$$G = \left[\frac{P_n^m(\mu)}{r^{n+1}} + \frac{(-1)^n}{(n-m)!} \oint_0^\infty \frac{k+k_0}{k-k_0}\, e^{-k(z+f)} k^n J_m(kR)\, dk\right] \cos m\phi + 2\pi i k_0^{n+1} e^{-k_0(z+f)} J_n(k_0 r) \cos m\phi, \tag{5.1}$$

where $r^2 = x^2 + y^2$, $R^2 = z^2 + r^2$ and $n \geqslant m$. The integral is taken as a Cauchy principal value integral. The path of integration is chosen such that the point $(0, 0, f)$ acts only as a source.

When $f \neq 0$ (multipole is situated below the mean free surface), the integral in (5.1) is regular at $r = 0$. Therefore the singularity of G is contained in the first term involving the associated Legendre polynomial. This argument, however, is not correct when the multipole lies on the free surface, $f = 0$. In this case the integral is not regular at the origin. See equation (3) of Havelock (1955) for an expansion of the integral in (5.1) from which a proof of the above statements can be readily deduced. It is not necessary to consider arbitrary (positive) n and m in (5.1), but only $n = m$ for $f = 0$. Call the integral in (5.1) F_n and take the difference $F_{n+2} - k_0^2 F_n$; it is simple to prove that this difference contains a singularity. In this manner the entire hierachy of singularities can be generated. It is thus necessary to consider only the special case $n = m$ in evaluating (5.1) rather than dealing with an arbitrary n and m where $n \geqslant m$. Consequently the double infinity of values (m, n) is reduced to the single infinity of values (n).

The author has proved that the multipole G can be written as

$$G(a, \theta, 0) = [G_1^{(n)}(a, \theta) + iG_2^{(n)}(a, \theta)] \cos n\phi, \tag{5.2}$$

where

$$\begin{aligned} G_1^{(n)} = {} & \frac{P_n^n(\mu)}{a^{n+1}} + \frac{(2n)!}{n!}\frac{(\sin\theta)^n}{a^{n+1}} - 2\pi k_0^{n+1} e^{-\beta\cos\theta}\, Y_n(\beta\sin\theta) \\ & - \Gamma(\tfrac{1}{2} - n)\, \Gamma(\tfrac{1}{2} + n)\, k_0^{n+1}\, e^{-\beta\cos\theta}\, [H_{-n}(\beta\sin\theta) - Y_{-n}(\beta\sin\theta)] \\ & - \frac{4}{\pi^{\frac{1}{2}}}\, \Gamma(\tfrac{1}{2} + n)\, (\tfrac{1}{2}k_0)^{n+1}\, (\tan\theta)^n\, (\sec\theta)^n\, e^{-\beta\cos\theta} \int_0^1 \frac{e^{t\beta\cos\theta}\, dt}{(t^2 + \tan^2\theta)^{n+\frac{1}{2}}}, \end{aligned} \tag{5.3}$$

and

$$G_2^{(n)} = 2\pi k_0^{n+1}\, e^{-\beta\cos\theta}\, J_n(\beta\sin\theta). \tag{5.4}$$

Here $H_{-n}(x)$ is the Struve function of order $(-n)$. Since the rather intricate analysis required to obtain (5.2) is not germane to the paper we omit it. Full details may be found in the author's internal report or in a forthcoming paper devoted to the evaluation of the three-dimensional multipole.

In the important case when $n = 0$ (point source) the above expressions reduce to

$$\begin{aligned} aG_1(a, \theta) = {} & 2 - \pi\beta\, e^{-\beta\cos\theta}\, [H_0(\beta\sin\theta) + Y_0(\beta\sin\theta)] \\ & - 2\beta\, e^{-\beta\cos\theta} \int_0^1 (t^2 + \tan^2\theta)^{-\frac{1}{2}}\, e^{t\beta\cos\theta}\, dt, \end{aligned} \tag{5.5}$$

and

$$aG_2(a, \theta) = 2\pi\beta\, e^{-\beta\cos\theta}\, J_0(\beta\sin\theta). \tag{5.6}$$

The integral in the multipole becomes infinite as θ approaches zero; therefore, a different approach, described in Appendix I of the internal report, is necessary for this case.

Only the point source ($n = 0$) is needed for the discussion of the forced heaving part of the problem; however, the diffraction problem requires the general multipole ($n \neq 0$).

6. Forced heave problem—formal solution†

We take as the forced heave potential

$$W^f = \sum_{n=1}^{\infty} a^{2n+2}(A_n' + iB_n')\, M_{2n}^{(0)}(\mu) + (S_1 + iS_2)\,(G_1 + iG_2), \tag{6.1}$$

where $S_1 + iS_2$ is the complex source strength, and $G_1 + iG_2$ is the velocity potential of the point source. The function $M_{2n}^{(0)}$ is the wave-free potential defined in § 4. Since the motion is symmetrical with respect to ϕ only terms involving $m = 0$

† This section is essentially equivalent to Havelock's work (1955) as regards the evaluation of the expansion coefficients and source strengths.

are needed, consequently the anti-symmetric wave-free potential N_{2n}^{2m-1} does not enter in the forced heave problem. The point source G is included because W^f must act as an outgoing progressive wave and hence satisfy the radiation condition. It should be emphasized that the expansion of W^f into an infinite series of the wave-free potentials alone is not complete and every attempt to compute the expansion coefficients will lead to different values simply because the expansion of a function into an incomplete set is not unique. When $\beta = 0$ the expansion (6.1) reduces to a Legendre polynomial expansion which is complete. We can look upon (6.1) as a generalization of the Legendre expansion.

The unknowns are the expansion coefficients A_n', B_n' $(n = 1, \ldots, \infty)$ and the source strengths S_1, S_2. All the unknowns are functions of β. Our problem is to determine these unknowns.

The forced potential W^f has dimensions of L^2T^{-1} so that S has dimensions L^2T^{-1} and W^S has L^{-1}. The unknowns A_n' and B_n' are not dimensionless but have the dimensions of a velocity LT^{-1}.

Now
$$a^2\frac{\partial W^f}{\partial r} = (S_1 + iS_2)\left[a^2\frac{\partial^2 G_1}{\partial r} + ia^2\frac{\partial G_2}{\partial r}\right] - \sum_{n=1}^{\infty}[(2n+1)P_{2n}(\mu) + \beta P_{2n-1}(\mu)]\,a^2(A_n' + iB_n'). \quad (6.2)$$

It is understood that r is to be equated to a. Let $\eta' = \eta_0' e^{-i\omega t}$ represent the forced vertical motion, then the kinematic boundary condition becomes

$$a^2\,\partial W^f/\partial r = i\omega\eta_0' a^2 P_1(\mu) \quad (r = a). \quad (6.3)$$

Dividing both sides of (6.3) by $-\omega\eta_0'$ and separating real and imaginary parts, we find

$$\left[\mathscr{S}_1 a^2\frac{\partial G_1}{\partial r} - \mathscr{S}_2 a^2\frac{\partial G_2}{\partial r}\right] - \sum_{n=1}^{\infty}[(2n+1)P_{2n}(\mu) + \beta P_{2n-1}(\mu)]\,A_n = 0, \quad (6.4)$$

and
$$\left[\mathscr{S}_1 a^2\frac{\partial G_2}{\partial r} + \mathscr{S}_2 a^2\frac{\partial G_1}{\partial r}\right] - \sum_{n=1}^{\infty}[(2n+1)P_{2n}(\mu) + \beta P_{2n-1}(\mu)]\,B_n = -P_1(\mu), \quad (6.5)$$

where
$$\mathscr{S}_1 = \frac{S_1}{a^2\omega\eta_0'}, \quad \mathscr{S}_2 = \frac{S_2}{a^2\omega\eta_0'}, \quad A_n = \frac{A_n'}{\omega\eta_0'}, \quad B_n = \frac{B_n'}{\omega\eta_0'}.$$

The unknowns $\mathscr{S}_1, \mathscr{S}_2, A_n, B_n$ $(m = 1, \ldots, \infty)$ are now dimensionless.

The basic equations (6.4) and (6.5) can be solved by using Rayleigh's integral to generate a linear system of equations where the number of variables and equations is infinite. By truncating the infinite system we can obtain approximate answers, the degree of approximation depending upon the accuracy required.

Multiply both sides of (6.4) and (6.5) by

$$\left.\begin{array}{ll} P_0(\mu) \quad \text{when} \quad j = 0, & \\ (2j+1)P_{2j}(\mu) + \beta P_{2j-1}(\mu) \quad \text{when} \quad j = 1, 2, \ldots, \infty, & \end{array}\right\} \quad (6.6)$$

and integrate over μ from 0 to 1. Use of Rayleigh's integral

$$\int_0^1 P_{2r}(\mu)\,P_{2s+1}(\mu)\,d\mu = \frac{(-1)^{r+s+1}(2r)!\,(2s+1)!}{4^{r+s}(2r-2s-1)\,(2r+2s+2)\,(r!)^2\,(s!)^2}, \quad (6.7)$$

will generate one equation with an infinite number of A_n or B_n for each value of j.

The set of equations obtained by the indicated integration of (6.4) and (6.5) is truncated to include the first four A_n and B_n. Thus there are 10 inhomogeneous equations in 10 unknowns: $\mathscr{S}_1, \mathscr{S}_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$.† There is, of course, one 10×10 system for each specified value of β.

As is common in diffraction problems the infinite system of coefficient equations is easily solved only in the low-frequency region ($\beta \leqslant 3$). In the intermediate frequency region ($3 \leqslant \beta \leqslant 10$) the system of equations becomes progressively more ill-conditioned as β increases and a large number of coefficients must be taken into account. However, in the high-frequency region ($\beta > 10$) one can use the asymptotic properties of the source term G which results in a considerable simplification in the structure of the equations, in that one of the coefficients is of an order of magnitude larger than the others. We can show directly from the infinite equation system that as $\beta \to \infty$,

$$B_1 \sim O(\beta^{-1}), \quad B_{n+1} \sim O(\beta^{-2}), \quad A_n \sim O(\beta^{-2}) \quad (n = 1, 2, \ldots), \tag{6.8}$$

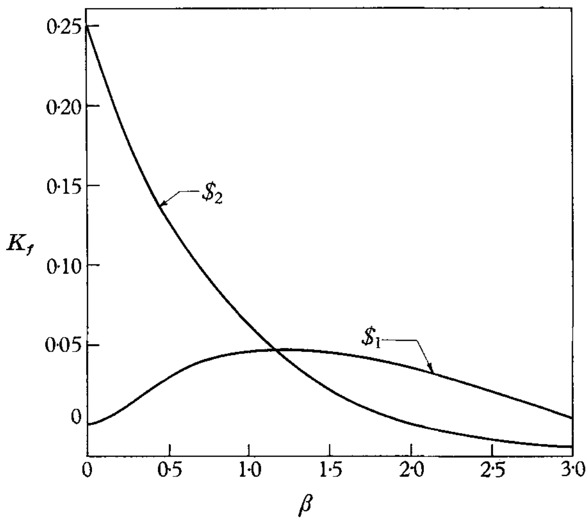

FIGURE 1. The real and imaginary parts of the complex source strength for forced heave.

provided that $\mathscr{S}_1$, $\mathscr{S}_2 \sim O(\beta^{-2})$, which will be so, since the source term G vanishes for large β. This is consistent with the solution at $\beta = \infty$ where the free surface boundary condition becomes

$$W = 0 \quad (\text{at } z = 0). \tag{6.9}$$

The source strengths $\mathscr{S}_1$ and $\mathscr{S}_2$ are computed using a 10×10 system and are shown in figure 1. The fact that $\mathscr{S}_2$ does not vanish at $\beta = 0$ is the reason why the added mass coefficient K_f (see § 7) does not vanish at $\beta = 0$. The calculations have been carried out to $\beta = 3$ (the low-frequency region); however, it is essential that computations be carried out to at least $\beta = 10$ in order to answer certain questions

† The explicit form of the simultaneous equations is given in the internal report.

pertaining to the asymptotic behaviour of the added mass and damping coefficient. Calculations in this frequency range are in progress.

This work is for *infinite* depth; the author has also considered this problem for *finite* depth. The result will be reported in the near future.

7. Added mass, damping, and wave-making coefficients

We now pass to a detailed consideration of a number of dynamical quantities of interest.

The steady-state vertical force on the sphere is

$$F_0^f = i2\pi\rho\omega a^2 \int_0^{\frac{1}{2}\pi} w^f \sin\theta \cos\theta \, d\theta, \tag{7.1}$$

where $F^f = F_0^f e^{-i\omega t}$. The forced heave potential w^f is given by (6.1) and the integration is at $r = a$ since the integral is over the immersed surface of the sphere. Instead of F_0^f, we use the dimensionless force

$$Z_0^f = \frac{F_0^f}{\omega^2\rho a^3 \eta_0'} = Z_1^f + iZ_2^f. \tag{7.2}$$

Substitution of (6.1) into (7.1) and use of (6.7) yields

$$-Z_1^f = \mathscr{S}_1 L_2 + \mathscr{S}_2 L_1 + (\tfrac{1}{8} + \tfrac{1}{6}\beta) B_1 - \tfrac{1}{48} B_2 + \tfrac{1}{128} B_3 - \tfrac{1}{256} B_4 + \ldots, \tag{7.3}$$

$$Z_2^f = \mathscr{S}_1 L_1 - \mathscr{S}_2 L_2 + (\tfrac{1}{8} + \tfrac{1}{6}\beta) A_1 - \tfrac{1}{48} A_2 + \tfrac{1}{128} A_3 - \tfrac{1}{256} A_4 + \ldots. \tag{7.4}$$

Here $$L_1(\beta) = \int_0^{\frac{1}{2}\pi} (aG_1) \sin\theta \cos\theta \, d\theta; \quad L_2(\beta) = \int_0^{\frac{1}{2}\pi} (aG_2) \sin\theta \cos\theta \, d\theta. \tag{7.5}$$

Trivial calculations show that the vertical force component Z_1^f is π radians out of phase with the acceleration, while Z_2^f is in phase with the vertical velocity. Following Ursell (1949, 1957) and Havelock (1955) we introduce the added mass coefficient K_f and the damping coefficient H_f which are directly related to Z_1^f and Z_2^f. Define

$$K_f = \frac{\text{force component } \pi \text{ radians out of phase with acceleration}}{\text{acceleration} \times \text{mass of displaced fluid}},$$

$$H_f = \frac{\text{force component in phase with vertical velocity}}{\text{acceleration} \times \text{mass of displaced fluid}}.$$

Then $$K_f = -3Z_1^f; \quad H_f = 3Z_2^f. \tag{7.6}$$

The computed values of K_f and H_f are shown in figures 2 and 3. Our values are in fair agreement with those of Havelock (1955) for $\beta \leqslant 0{\cdot}8$, although above these values the values are somewhat different. The discrepancy is partly due to our using a 10×10 system rather than his 8×8 system as well as his use of power-series expansions for the evaluation of the source term rather than numerical quadrature.

At $\beta = 0$ there is a discrepancy between the behaviour of K_f as obtained by Havelock and by the author. According to Havelock (see figure 1 of his paper), K_f has infinite *positive* slope at $\beta = 0$, and hence the values of K_f at $\beta = 0{\cdot}1$, $0{\cdot}2$

are greater than $K_f(0) = 0{\cdot}8305$.† In contrast to Havelock's result the slope of K_f is now *negative* infinite with a limiting value of $K_f(0) = 0{\cdot}8309$. In fact it can be shown that the slope is proportional to $\log \beta$ so that the effect of the infinite slope is confined to a very small region around $\beta = 0$. It would seem from an examination of Havelock's diagram that the slope of K_f behaves at least as β^{-1}

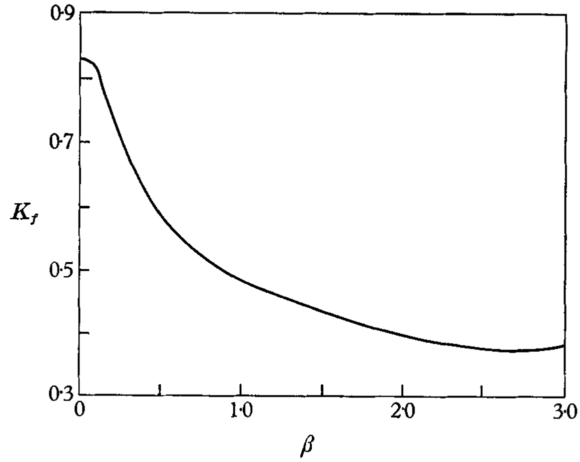

FIGURE 2. The added mass coefficient K_f in forced heave as a function of β.

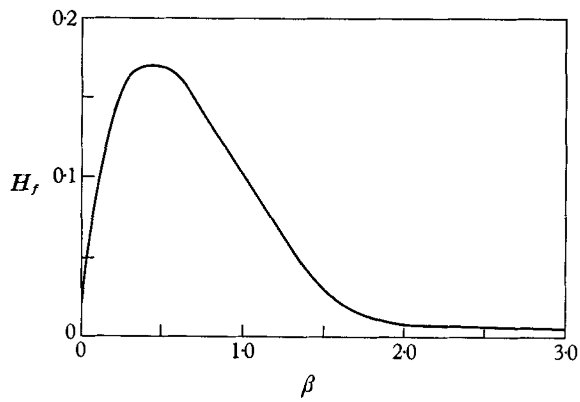

FIGURE 3. The damping coefficient H_f in forced heave as a function of β.

at $\beta \sim 0$. An examination of the numerical values of $\mathscr{B}_2$ and B_1 (the two largest numerical terms) at $\beta = 0$ and $\beta = 0{\cdot}1$ shows that both functions are numerically smaller at $\beta = 0{\cdot}1$. It is fairly simple to show that $K_f(0) > K_f(\beta)$ for $(0 < \beta \leqslant 0{\cdot}1)$, and in turn $K_f(0{\cdot}1) > K_f(0{\cdot}2)$, etc. The results of Havelock are probably due to a mistake in numerical computations.

The high-frequency asymptotics of K_f and H_f have a very simple structure. As we remarked in § 6 only B_1 is $O(\beta^{-1})$ all other coefficients and source strengths being $O(\beta^{-2})$. Thus directly from (7.3), (7.4) and (7.6) we have

$$K_f \sim \tfrac{1}{2} + O(\beta^{-1}); \quad H_f \sim O(\beta^{-2}). \tag{7.7}$$

† The original value given by Havelock, $K_f = 0{\cdot}828$ is incorrect. A recomputation using his values of the source strengths and expansion coefficients yields the above value.

Actually Ursell (1957) has stated (although no proof is given)† that the high-frequency asymptotic forms are

$$K_f \sim \tfrac{1}{2} - \frac{3}{16\beta} + \dots; \quad H_f \sim \frac{27}{4\beta^4} + \dots. \tag{7.8}$$

Unfortunately these results do not bridge the gap between the computations which at present only go up to $\beta = 3$. Therefore it is imperative to carry the numerical computations to $\beta = 10$ at which point it is believed that the asymptotic series (7.8) and the numerical results should join smoothly. For example, K_f at $\beta = 3{\cdot}0$ is 0·3772 (numerical computation) and 0·4375 (asymptotic expansion) so that the discrepancy is not small enough to neglect.

Unlike the two-dimensional problem studied by Ursell the added mass coefficient is finite at $\beta = 0$ and $\beta = \infty$. The behaviour of K_f is rather complicated in the low-frequency region where it varies by approximately a factor of two and then asymptotically approaches its limiting value of 0·5. The damping coefficient also undergoes an unusually complicated behaviour in the low-frequency region before rapidly decaying to its limiting value zero. Now H_f vanishes at the two extremes $\beta = 0$ and $\beta = \infty$, as of course it must as these two degenerate boundary conditions preclude wave propagation.

The standing waves (represented by the wave-free potentials) in the immediate vicinity of the sphere interacting with the sphere are the direct cause of the added mass. This is the reason why attempts utilizing only sources to determine the added mass are incomplete, although they are successful in predicting the damping coefficient which is directly related to the transport of energy to infinity by diverging progressive waves.

In the high-frequency region very little energy is propagated to infinity and most of the wave motion is in the form of standing waves in the vicinity of the sphere. These standing waves are not very sensitive to β and we would expect that the added mass coefficient would be slowly varying. In this respect the high-frequency approximation to the free surface-boundary condition (i.e $W = 0$ at $z = 0$) is probably a very good engineering approximation in the sense that the added mass coefficient varies so slowly that its frequency dependence may be neglected to first order and the added mass now becomes a function only of the geometry of the body rather than a function of both geometry and frequency as in the more rigorous theory.

The wave amplitude ζ at the mean free surface is given by

$$\zeta = \frac{1}{g}\frac{\partial\psi}{\partial t}\bigg|_{\mu=0} = -i\frac{\omega}{g}W_f\bigg|_{\mu=0} = \zeta_1 + i\zeta_2. \tag{7.9}$$

Using (8.3) and recalling that $P_{2n-1}(0) = 0$ we find

$$\left.\begin{aligned}\Gamma_1 &= \frac{\zeta_1}{\eta_0} = \beta[\mathscr{S}_1 aG_2 + \mathscr{S}_2 aG_1] + \beta\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{2n+1} P_{2n}(0)\,B_n,\\ \Gamma_2 &= \frac{\zeta_2}{\eta_0} = -\beta[\mathscr{S}_1 aG_1 - \mathscr{S}_2 aG_2] - \beta\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n+1} P_{2n}(0)\,A_n,\end{aligned}\right\} \tag{7.10}$$

where the source terms are evaluated at $\mu = 0$.

† Dr Ursell has kindly furnished the author with an outline of the proof in some private correspondence.

Both Γ_1 and Γ_2 are highly oscillatory functions. Instead of studying them directly we introduce a smooth function which we call the wave-making coefficient

$$U(\beta) = \text{amplitude of wave height at } r = a = [\Gamma_1^2 + \Gamma_2^2]^{\frac{1}{2}}. \tag{7.11}$$

(This is *not* the same function which Ursell terms the wave-making coefficient in his two-dimensional studies.) The calculations are summarized in figure 4. The asymptotic limits are

$$\left.\begin{aligned} U(\beta) &\sim 0 \quad &(\beta \sim 0),\\ U(\beta) &\sim O(1) \quad &(\beta \sim \infty).\end{aligned}\right\} \tag{7.12}$$

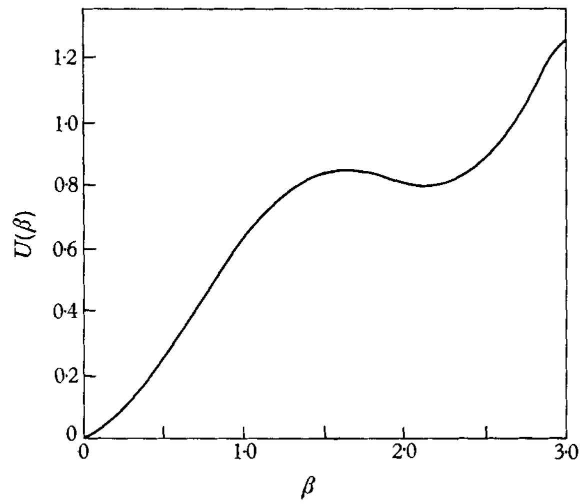

FIGURE 4. The wave-making coefficient in forced heave.

8. Diffraction problem

We now pass to the diffraction problem. The boundary condition (C) of § 2 is used to determine the diffracted wave potential W^d in terms of W^i. The incident wave potential in spherical co-ordinates is

$$W^i(r, \theta, \phi) = \frac{g\sigma}{\omega} \exp(-k_0 r \cos\theta) \exp(ik_0 r \sin\theta \cos\phi). \tag{8.1}$$

The diffracted wave potential is to be expanded in the series

$$W^d(r,\theta,\phi) = \frac{g\sigma}{\omega} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} a^{2n+2}(C_n^m + iD_n^m) M_{2n}^{2m} + \frac{ig\sigma}{\omega} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} a^{2n+2}(E_n^m + iL_n^m) N_{2n}^{2m-1}$$
$$+ \frac{g\sigma}{\omega} \sum_{n=1}^{\infty} (S_1^{(n)} + iS_n^{(n)}) (aG_1^{(n)} + iaG_2^{(n)}) \cos n\phi. \tag{8.2}$$

The functions M_{2n}^{2m} and N_{2n}^{2m-1} are the wave-free potentials defined in § 4. The unknown expansion coefficients $C_n^m, D_n^m, E_n^m, L_n^m$ are dimensionless, as are the source strengths $S_1^{(n)}$ and $S_2^{(n)}$. The multipole series represents the continuous spectrum and need be summed only over n rather than over n and m by virtue of the remarks made at the beginning of § 5. The multipole series is added to make W^d behave as an outgoing progressive wave and the infinite series is needed because each partial wave corresponding to a fixed n must satisfy the radiation condition.

Upon differentiating (8.1) and (8.2) with respect to r, using the boundary condition (C) and separating the resultant equation into its real and imaginary parts, we obtain for the real part

$$\beta e^{-\beta\cos\theta}\Psi_1(\beta,\theta,\phi) = -\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\left[(2n+1)P_{2n}^{2m}(\mu)+\frac{n\beta}{(n-m)}P_{2n-1}^{2m}(\mu)\right]C_n^m\cos 2m\phi$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[(2n+2)P_{2n+1}^{2m-1}(\mu)+\frac{(2n+1)\beta}{2(n-m+1)}P_{2n}^{2m-1}(\mu)\right]L_n^m\cos(2m-1)\phi$$
$$+\sum_{n=0}^{\infty}\left[S_1^{(n)}a^2\frac{\partial G_1^{(n)}}{\partial r}-S_2^{(n)}a^2\frac{\partial G_2^{(n)}}{\partial r}\right]\cos n\phi, \tag{8.3}$$

where

$$\Psi_1(\beta,\theta,\phi) = \cos\theta\cos(\beta\sin\theta\cos\phi)+\sin\theta\cos\phi\sin(\beta\sin\theta\cos\phi). \tag{8.4}$$

The double summation is reduced to a single summation by using the orthogonal properties of the cosine. Multiply both sides of (8.7) by $\cos 2j\phi\,d\phi$ and integrate from 0 to $\frac{1}{2}\pi$. As a result of the integration the series containing L_n^m vanishes leaving

$$\frac{4}{\pi}e^{-\beta\cos\theta}\int_0^{\frac{1}{2}\pi}\Psi_1(\beta,\theta,\phi)\cos 2j\phi\,d\phi$$
$$= -\sum_{n=j+1}^{\infty}\left[(2n+1)P_{2n}^{2m}(\mu)+\frac{n\beta}{(n-m)}P_{2n-1}^{2m}(\mu)\right]C_n^m$$
$$+\left[S_1^{(n)}a^2\frac{\partial G_1^{(2n)}}{\partial r}-S_2^{(n)}a^2\frac{\partial G_2^{(2n)}}{\partial r}\right]. \tag{8.5}$$

The integral on the left-hand side can be written explicitly in terms of Bessel functions. Finally

$$(-1)^j e^{-\beta\cos\theta}[2\cos\theta J_{2j}(\beta\sin\theta)+\sin\theta J_{2j+1}(\beta\sin\theta)-\sin\theta J_{2j-1}(\beta\sin\theta)]$$
$$= -\sum_{n=j+1}^{\infty}\left[(2n+1)P_{2n}^{2m}(\mu)\frac{n\beta}{(n-m)}O_{2n-1}^{2m}(\mu)\right]C_n^m$$
$$+\left[S_1^{(2n)}a^2\frac{\partial G_1^{(2n)}}{\partial r}-S_2^{(2n)}a^2\frac{\partial G_2^{(2n)}}{\partial r}\right]. \tag{8.6}$$

This is the first of four basic equations. In like manner we take the imaginary part of boundary equation and repeat the above procedure. The final result is

$$-\sum_{n=j+1}^{\infty}\left[(2n+1)P_{2n}^{2m}(\mu)+\frac{n\beta}{(n-m)}P_{2n-1}^{2m}(\mu)\right]D_n^m$$
$$+\left[S_1^{(2n)}a^2\frac{\partial G_2^{(2n)}}{\partial r}+S_2^{(2n)}a^2\frac{\partial G_1^{(2n)}}{\partial r}\right]=0. \tag{8.7}$$

Here the E_n^m series vanished.

The two basic equations (8.6) and (8.7) are coupled by the occurrence of S_1 and S_2 in each. Two more basic equations can be obtained which relate E_n^m and L_n^m by using the cosine orthogonalization procedure with $\cos(2j-1)\phi$.

We can use the orthogonalization procedure described in § 6 to generate an infinite system of equations where the functions are

$$(2n+1)\,P_{2n}^{2m}(\mu)+\frac{n\beta}{(n-m)}\,P_{2n-1}^{2m}(\mu) \tag{8.8}$$

for the equations connecting S_1, S_2, D_n^m and C_n^m; and

$$(2n+2)\,P_{2n+1}^{2m-1}(\mu)+\frac{(2n+1)\,\beta}{2(n-m+1)}\,P_{2n}^{2m-1}(\mu) \tag{8.9}$$

for the equations connecting S_1, S_2, E_n^m and L_n^m. For each value of m there exists an infinite equation system.

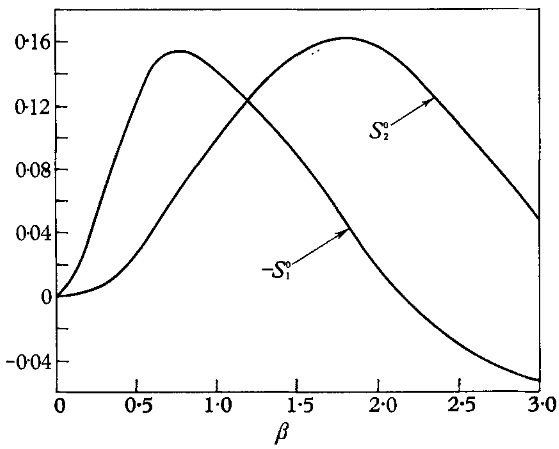

FIGURE 5. The real and imaginary parts of the complex source strength for the diffraction problem.

For the purposes of this paper the main interest lies in the evaluation of the vertical force on the sphere due to the incident wave and not in the resultant diffraction pattern. This allows a considerable simplification in the analysis. The terms of W^d containing $\cos{(2m-1)}\,\phi$ do not contribute to the vertical force because they are anti-symmetric and hence vanish when integrated over the surface of the sphere. The only term in $\cos 2m\phi$ which contributes to the vertical force is $m = 0$, therefore only the $m = 0$ case is needed and we merely set $m = 0$ in (8.6) and (8.7). The resultant equations are similar to the equations that appeared in the forced heave problem. Upon carrying out the same procedure we finally obtain the required coefficient equations which are then truncated to 10×10 and solved by Gauss elimination.

The behaviour of the source strengths S_1^0 and S_2^0 are shown in figure 5. Unlike the forced heave case, here both source strengths vanish at $\beta = 0$.

9. Force on fixed sphere

The vertical force on the fixed sphere is obtainable from the total potential $(W^i + W^d)$. For convenience the forces due to the incident wave and diffracted wave are evaluated separately.

The force due to the incident wave potential W^i is

$$Z_0^i = \frac{F_0^i}{a^2\rho g\sigma} = i2\pi \int_0^{\frac{1}{2}\pi} e^{-\beta\cos\theta} J_0(\beta\sin\theta)\sin\theta\cos\theta\, d\theta. \tag{9.1}$$

The incident wave potential W^i and the vertical force Z_0^i are $\frac{1}{2}\pi$ out of phase with each other.

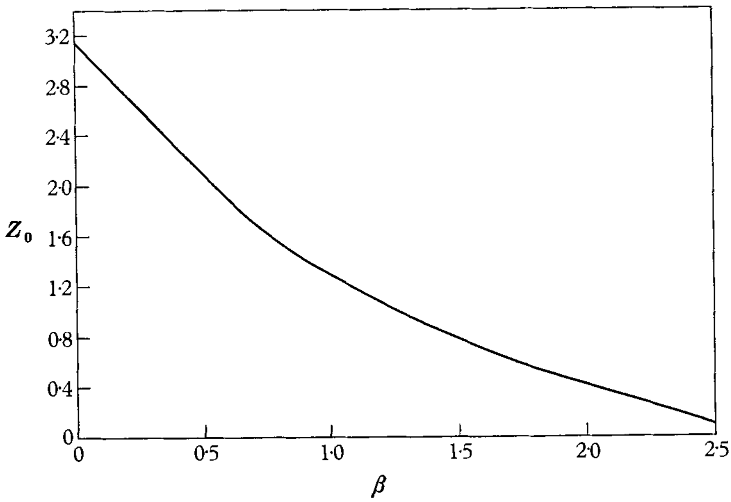

FIGURE 6. The total vertical force on a fixed sphere as a function of β.

The force on the sphere caused by the diffracted wave potential is

$$F_0^d = i\rho\omega a^2 \int_0^{2\pi}\int_0^{\frac{1}{2}\pi} W^d \sin\theta\cos\theta\, d\theta\, d\phi, \tag{9.2}$$

where W^d is given by (8.5) and the integration is to be taken over ϕ first. It is simple to show that the N_{2n}^{2m-1} multipole series vanishes completely upon integration. The M_{2n}^{2m} series vanishes for all values of m except $m = 0$ while the multipole series degenerates to only the source term ($n = 0$).

In terms of the dimensionless force Z_0^d

$$Z_0^d = \frac{F_0^d}{a^2\rho g\sigma} = Z_1^d + iZ_2^d, \tag{9.3}$$

we have

$$\left.\begin{aligned} -Z_1^d &= S_1^0 L_2 + S_2^0 L_1 + (\tfrac{1}{8}+\tfrac{1}{6}\beta)\, C_1^0 - \tfrac{1}{48} C_2^0 + \tfrac{1}{128} C_3^0 - \tfrac{1}{256} C_4^0 + \dots, \\ Z_2^d &= S_1^0 L_1 - S_2^0 L_2 + (\tfrac{1}{8}+\tfrac{1}{6}\beta)\, D_1^0 - \tfrac{1}{48} D_2^0 + \tfrac{1}{128} D_3^0 - \tfrac{1}{256} D_4^0 + \dots. \end{aligned}\right\} \tag{9.4}$$

The L_1 and L_2 functions have already been defined by (7.5).

The total force on the sphere is simply

$$Z_0 = Z_1^d + i[Z_2^d + Z_0^i] = |Z_0|\, e^{i\delta}. \tag{9.5}$$

Figure 6 illustrates the behaviour of the total force $|Z_0|$ as a function of β.

The contribution to the total force from the diffracted wave components is relatively small.† The total force is largest in the long-wave region (as we would expect) and rapidly decreases as β increases.

† In the internal report there are graphs of the various wave components.

10. Amplitude and phase angle in free heave

We are now in a position to effect the synthesis of both solutions through the equation of motion (3.1). Substitution of (3.3) into (3.1) yields the basic equation

$$C\eta_0 e^{i\epsilon} = \sigma[iZ_2^i + Z_1^d + iZ_2^d] + \eta_0[Z_1^f + iZ_2^f]\, e^{i\epsilon}, \tag{10.1}$$

where η_0' is identified as η_0 and $C = \pi(\frac{2}{3}\beta - 1)$. Upon setting $\Sigma = \eta_0/\sigma$ = relative heave amplitude and separating (10.1) into its real and imaginary parts, we have

$$[(C - Z_1^f)\cos\epsilon - Z_2^f \sin\epsilon]\,\Sigma = Z_1^d, \tag{10.2}$$

$$[(C - Z_1^f)\sin\epsilon + Z_2^f \cos\epsilon]\,\Sigma = Z_2^i + Z_2^d. \tag{10.3}$$

These two simultaneous equations are sufficient to determine the two unknowns ϵ and Σ. Eliminate Σ between these equations, obtaining a single equation in ϵ,

$$\tan\epsilon = \frac{(C - Z_1^f)(Z_2^i + Z_2^d) - Z_1^d Z_2^f}{(C - Z_1^f) Z_1^d + Z_2^f (Z_2^i + Z_2^d)}. \tag{10.4}$$

Once ϵ has been evaluated (at a specified β), it can be put back into (10.2) or (10.3) and Σ also computed. The results of the numerical calculations are shown in figures 7 and 8.

The results for the relative heave amplitude (figure 8) are similar to that of a driven harmonic oscillator with damping present. The amplitude starts at unity for $\beta \sim 0$, rapidly passes to a maximum at $\beta \approx 1{\cdot}5$ and then undergoes a monotonic decrease to zero as β becomes larger. Note the large value of Σ at resonance. The variation of the phase angle (figure 7) with respect to β is also interesting. In the vicinity of resonance ($\beta \approx 1{\cdot}5$) there is a rapid change in the phase angle of π radians. The speed with which ϵ changes at resonance is indicative of the small value of the term proportional to the heave velocity. If ϵ changed discontinuously, then the term proportional to the heave velocity would vanish.

We can rewrite (10.1) in the following form:

$$(\tfrac{2}{3}\pi\beta - Z_1^f)\,\Sigma + iZ_2^f\Sigma - \pi\Sigma = [Z_1^d + i(Z_2^i + Z_2^d)]\, e^{-i\epsilon}, \tag{10.5}$$

so that it is essentially in the form of a damped, driven harmonic oscillator

$$M\frac{d^2y}{dt^2} + N\frac{dy}{dt} + Ry = F_0 e^{-i\omega t}. \tag{10.6}$$

The second term on the left-hand side of (10.5) is imaginary and hence in phase with the heave velocity. Thus Z_2^f is proportional to the damping in free heave. Reference to (7.6) shows that Z_2^f is also proportional to the damping in forced heave. The first term of (10.5) is the term involving the mass and Z_1^f can be thought of as the added mass in free heave. It is, of course, also the added mass in forced heave. The right-hand side of (10.4) plays the part of a forcing function while the third term on the left-hand side is proportional to the water-plane area of the sphere.

An important result is that the added mass and damping coefficients in free heave are strongly frequency dependent and not even remotely constant as the Froude–Krylov theory postulates. In fact one would expect the Froude–Krylov theory to be approximately true in the low-frequency region since it is based upon plausible assumptions which at first glance would seem to hold for this frequency region. This expectation is not true as the added mass and damping coefficients change most rapidly in this frequency region.

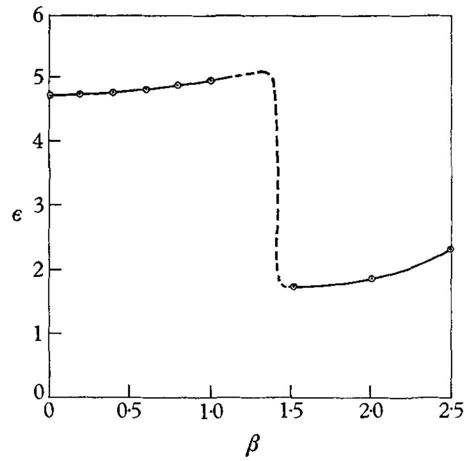

FIGURE 7. The phase angle ϵ as a function of β. ⊙, Computed values.

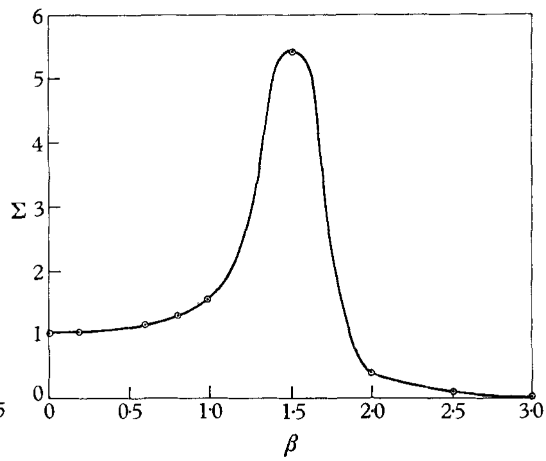

FIGURE 8. The relative free heave amplitude Σ as a function of β. ⊙, Computed values.

It is important that detailed experiments for both forced and free heave be undertaken on the sphere with the intent of verifying or disproving the analysis presented. Unfortunately the author (being in an optical laboratory) is not in a position to carry out these experiments and hopes that someone with the proper facilities can undertake the necessary investigations.

I am deeply indebted to Prof. F. Ursell for invaluable advice and discussions during the course of this investigation. I would also like to thank Drs R. C. MacCamy, J. V. Wehausen, and J. B. Keller for encouragement.

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